Published by Institute of Physics Publishing for SISSA/ISAS
Received:
Accepted:

Gauge/gravity duality, Green functions of $\mathcal{N}=2$ SYM and radial/energy-scale relation

JHEP10(2002)005

Xiao-Jun Wang

Institute of Theoretical Physics, Beijing, 100080, P.R. China
E-mail: wangxj@itp.ac.cn

Sen Hu

Department of Mathematics, University of Science and Technology of China
AnHui, HeFei, 230026, P.R. China
E-mail: shu@ustc.edu.cn

Abstract: We consider supergravity configuration of D5-branes wrapped on supersymmetric 2-cycles and use it to calculate one-point and two-point Green functions of some special operators in $\mathcal{N}=2$ super Yang-Mills theory. We show that Green functions obtained from supergravity include two very different parts. One of them corresponds to perturbative results of quantum field theory, and another is a non-perturbative effect which corresponds to contribution from instantons with fractional charge. Comparing Green functions obtained from supergravity and gauge theory, we obtain radial/energy-scale relation for this gauge/gravity correspondence with $\mathcal{N}=2$ supersymmetry. This relation leads right β-function of $\mathcal{N}=2$ SYM from supergravity configuration.

Keywords: D-brane, AdS/CFT correspondence, Super Yang-Mills theory.

http://jhep.sissa.it/archive/papers/jhep102002005/jhep102002005.pdf

Contents

1. Introduction

Recently, Maldacena's conjecture on AdS_5/CFT_4 correspondence [1] has been thoroughly investigated in large-N limit, and has been shown that it is a precise duality between the $\mathcal{N}=4$ super Yang-Mills (SYM) theory in 4-dimension and the type-IIB superstring in $AdS_5 \times S^5$. It makes more and more physicists believe that a lot of information of gauge theories can be obtained by studying their dual gravity background produced by stacks of D-branes even though supersymmetry and/or conformal invariance are (partly) broken.

The simplest extension of AdS_5/CFT_4 correspondence is to consider $\mathcal{N}=2$ SYM and its supergravity (SUGRA) duals. This dual SUGRA configurations have been studied in many literatures. Usually, there are two ways to reduce the number of preserved supercharges. One of them is to place a stack of D3-branes at the apex of an orbifold [2] or of a conifold [3]. Both for the orbifold and the conifold the conformal invariance can be naturally broken by means of fractional D3-branes [4]. Then one can realize a $\mathcal{N}=2$ non-conformal SYM theory in four dimensions. The recent reviews on this approach can be found in refs. [5]. Another method is to consider D-branes whose world-volume is partially wrapped on a supersymmetric cycle inside a K3 or a Calabi-Yau manifold. The unwrapped part of the brane world-volume remains flat and supports a gauge theory. Meanwhile, the normal bundle to the wrapped D-branes has to be partially twisted [6] in order to make some world-volume fields become massive and decouple. This method has been first used in ref. [7] to study pure $\mathcal{N}=1$ SYM theory in four dimensions, and later it has been generalized to study $\mathcal{N}=2$ SYM in four dimension [8, 9] and other cases with different space-time dimesions and different numbers of supersymmetry [10]. The purpose of this

JHEP10(2002)005

paper is to study some Green functions of dimension four operator[1] of $\mathcal{N} = 2$ SYM by means of wrapped D5 brane configuration, and to find radial/energy-scale relation of this gauge/gravity duality.

A crucial ingredient for the gauge/gravity correspondence is the relation between the radial parameter of the SUGRA solution and the energy scales of the gauge theory. In particular, authors of ref. [11] have established a formal relation between radial flow of 5-d gravity and renormalization running of 4-d gauge theory. However, in general it maybe ambigous and difficult to establish a exact radial/energy-scale relation for non-conformal theories [12]. Recently, for pure $\mathcal{N} = 2$ and $\mathcal{N} = 1$ SYM this difficulty has been partly overcome due to the work of Di Vecchia, Lerda and Merlatti (DLM) [13]. They studied β-function and chiral anomaly of pure $\mathcal{N} = 2$ SYM[2] via studying their dual SUGRA configurations which is constructed by D5-branes wrapped a supersymmetry 2-cycle. According to some naive arguments, they also established a simple radial/energy-scale relation for this gauge/gravity correspondence configuration. Their result should be right, perturbatively at least, since it leads to right β-function of $\mathcal{N} = 2$ SYM which can be obtained perturbative calculation of quantum field theory (QFT). However, It is still necessary to study this relation in different way. This is just one of main aims of this present paper.

Our idea is motivated by UV/IR relation [12] in gauge/gravity duality. Concretely, the near boundary limit (in most case, it is large radial limit), i.e., IR limit of SUGRA description corresponds to UV limit of dual SYM. This relation will explicitly exhibit in correlation functions (or Green function for non-conformal) resulted from SUGRA description and QFT method. In other words, if we calculate some Green functions of SYM in terms of perturbative QFT method, it will in general involve an UV cut-off. Meanwhile, when we calculate these Green functions via dual SUGRA description, it will in general involve an large radial cut-off. Therefore, we can investigate divergence behavior of Green functions resulted from SUGRA and QFT respectively, and compare these terms possessing same pole behavior. Then we will obtain an exact radial/energy-scale relation for this gauge/gravity correspondence configuration.

Since $\mathcal{N} = 2$ SYM is a non-conformal theory, its Green functions obtained from SUGRA description must appear some features that are very different from $\mathcal{N} = 4$ SYM. A direct feature is that they should include many non-perturbative effects because dual SUGRA describes a strong coupled $\mathcal{N} = 2$ SYM in fact. At QFT side, it has been well-known that these non-perturbative effects should be induced by instanton [14], but they can not be obtained via perturbative calculation of QFT method. In this paper, we will see that this effect can indeed be yielded from dual SUGRA configuration. In particular, the one-point function of dimension four operator obtained from SUGRA implies that gauge bosons and scalars will condensate, and it is instanton effect entirely.

The paper is organized as follows. In section 2 we review the $\mathcal{N} = 2$ SUGRA solution [8, 9, 13] in a first order formalism. It creates a background of D5-branes wrapped

[1] In this paper we focus on a non-conformal field theory. So that the dimension of operators means that mass dimension rather than conformal dimension

[2] The paper of DLM also includes study on β-function and chiral anomaly of $\mathcal{N} = 1$ SYM. However, in this paper we only focus on $\mathcal{N} = 2$ SYM due to the reason to be mentioned in last section.

JHEP10(2002)005

on supersymmetry 2-cycle. In section 3 we consider the fluctuations on this background, and find the solution of these fluctuation field whose boundary value couples to dimension four operator of $\mathcal{N}=2$ SYM. In section 4, the one-point and two-point Green functions of this operator are calculated in certain limit. Then we analysis the divergence behavior of the Green functions, and find radial/energy-scale relation for this gauge/gravity duality configuration. Some further results and β-function of $\mathcal{N}=2$ SYM are also achieved in this section. Finally, a brief summary is in section 5.

2. Wrapped D5-brane solution and $\mathcal{N}=2,\ D=4$ SYM

In this section we review wrapped D5-branes configuration which is dual to pure $\mathcal{N}=2$ SYM theory in four dimensions.

The essential idea is to consider the SO(4) gauged supergravity in $D=7$ [15], find its domain-walls solution which wraps on a 2-sphere, and then lift it up in ten dimensions [7, 16]. This SO(4) gauged SUGRA can be obtained via perform an S^3 reduction in type IIB SUGRA [17]. In order to obtain a solution dual to $\mathcal{N}=2$ SYM which preserves eight supercharges, one should truncate the SO(4) gauge group of 7-d SUGRA to its U(1) subgroup. The lagrangian for this truncated gauged SUGRA is

$$\mathcal{L}_7=\sqrt{-\det G}\left\{R-\frac{5}{16}\partial_i y\partial^i y-\partial_i x\partial^i x-\frac{1}{4}e^{-2x-y/2}F_{ij}F^{ij}-\right.$$
$$\left.-\frac{1}{12}e^{-y}H_{ijk}H^{ijk}+\frac{4e^{y/2}}{r_0^2}\right\}, \tag{2.1}$$

with

$$F_{ij}=\partial_i A_j-\partial_j A_i\,,\qquad H_{ijk}=\partial_i A^{(2)}_{jk}-\partial_j A^{(2)}_{ik}+\partial_k A^{(2)}_{ji}\,.$$

Here $i,j,k=0,\dots,6$ are 7-d space-time indices, r_0 is constant with length dimension, $x,\ y$ are scalars fields yielded from 10-d dilation and dimensional reduction, A_i is U(1) gauge field and $A^{(2)}$ are two-form potential inheriting from type IIB SUGRA.

The metric ansatz for the domain-wall solution is

$$ds_7^2=e^{2f(r)}(dx_{1,3}^2+dr^2)+r_0^2e^{2g(r)}d\Omega_2^2 \tag{2.2}$$

where $dx_{1,3}^2$ is the Minkowski metric on $\mathbf{R}_{1,3}$, r is the transverse coordinate to the domain-wall, and $d\Omega_2^2=d\theta^2+\sin^2\theta d\varphi^2$ $(0\le\theta\le\pi,\ 0\le\varphi\le 2\pi)$ is the metric of unit 2-sphere. To implement the topological twist that preserves eight supercharges, we have to identify the U(1) gauge field with the spin-connection on the tangent bundle to the sphere, i.e., $A=r_0^{-1}\cos\theta d\varphi$. It is also consistent to set $A^{(2)}=0$ and scalar fields $x,\ y$ are only dependent on r. Then to find the domain-wall of this truncated gauged SUGRA is just to solve the field equations of scalar functions $f(r)$, $g(r)$, $x(r)$ and $y(r)$.

Since all scalar functions are independent of θ and φ, one can further perform an S^2 reduction in this 7-d SUGRA, the result action is

$$S_5=\eta_5\int d^4xdr\ e^{2k}\left\{4\partial_\mu k\partial^\mu k-2\partial_\mu h\partial^\mu h-\partial_\mu x\partial^\mu x+\right.$$
$$\left.+4\left(\frac{dk}{dr}\right)^2-2\left(\frac{dh}{dr}\right)^2-\left(\frac{dx}{dr}\right)^2-V(x,h)\right\}, \tag{2.3}$$

where we have imposed the relation $y = -4f$ [8] such that

$$h = g - f\,, \qquad k = \frac{3}{2}f + g\,,$$
$$V(x,h) = -r_0^{-2}\left(4 + 2e^{-2h} - \frac{1}{2}e^{-4h-2x}\right), \tag{2.4}$$

η_5 is normalization constant which in principle has 10-d origin, and 4-d space-time indices μ, ν are raised or lowered by Minkowski metric.

Then the domain-wall solution is conveniently represented by the variable $z = e^{2h}$

$$e^{2k+x} = ze^{2z}\,, \qquad e^{-2x} = 1 - \frac{1+ce^{-2z}}{2z}\,, \tag{2.5}$$

where c is integration constant. This solution satisfies the following first-order Hamilton equations

$$\frac{dk}{dr} = \frac{1}{4}\mathcal{W}\,, \qquad \frac{dh}{dr} = -\frac{1}{4}\frac{\partial\mathcal{W}}{\partial h}\,, \qquad \frac{dx}{dr} = -\frac{1}{2}\frac{\partial\mathcal{W}}{\partial x}\,, \tag{2.6}$$

with

$$\mathcal{W}(x,h) = -r_0^{-1}(4\cosh x + e^{-2h-x})\,. \tag{2.7}$$

The following steps are substituting solution (2.5) such that f, g into 7-d metric (2.2), and up-lift it to ten diemensions [8, 17]. It describes the NS5 brane configuration. So that we can obtain the D5-brane configuration via performing S-duality [8, 13]. The 10-d metric in string frame is given by

$$ds_{10}^2 = e^{\omega}\Big\{dx_{1,3}^2 + zr_0^2 d\Omega_2^2 + e^{2x}r_0^2 dz^2 + r_0^2 d\tilde{\theta}^2 + \\ + \frac{r_0^2}{\Delta}(e^{-x}\cos^2\tilde{\theta}(d\tilde{\varphi}_1^2 + \cos\theta d\varphi)^2 + e^{x}\sin^2\tilde{\theta}d\tilde{\varphi}_2^2)\Big\}\,, \tag{2.8}$$

with

$$\Delta = e^{x}\cos^2\tilde{\theta} + e^{-x}\sin^2\tilde{\theta}\,, \tag{2.9}$$

and 10-d dilation given by

$$e^{2\omega} = e^{2z}\left(1 - \sin^2\tilde{\theta}\frac{1+ce^{-2z}}{2z}\right). \tag{2.10}$$

In addition, a magnetic R-R 2-form is present. It determines the parameter r_0 in terms of the number of wrapped D5-branes, N, and string parameters

$$r_0^2 = Ng_s\alpha'\,. \tag{2.11}$$

In order to make the structure of D5-branes clearer, one can define $H = e^{-2\omega}$ and [19]

$$\rho = r_0\sin\tilde{\theta}e^{z}, \qquad \sigma = r_0\sqrt{z}\cos\tilde{\theta}e^{z-x}\,. \tag{2.12}$$

Such that we have

$$ds_{10}^2 = H^{-1/2}(dx_{1,3}^2 + zr_0^2 d\Omega_2^2) + H^{1/2}(d\rho^2 + \rho^2 d\tilde{\varphi}_2^2) + \\ + \frac{H^{1/2}}{z}\{d\sigma^2 + \sigma^2(d\tilde{\varphi}_1 + \cos\theta d\varphi)^2)\} \,. \qquad (2.13)$$

It should be noted this solution does not hold maximal supersymmetry, but only preserves eight independent Killing spinors η determined by the following projection equation [8]

$$\eta = e^{-\frac{1}{2}(\tilde{\varphi}_1\Gamma^{67}+\tilde{\varphi}_2\Gamma^{89})}\eta_0 \,, \\ \Gamma^{6789}\eta_0 = -\tau_3\eta_0 \,, \qquad \Gamma^{4567}\eta_0 = -\eta_0 \,,$$

where η_0 is a constant spinor, τ_3 is the third Pauli matrix and $\Gamma^{\mu_1\cdots\mu_i} = \Gamma^{\mu_1}\cdots\Gamma^{\mu_i}$. The first projection is the same as that for fivebrane with tangent directions $\{0,1,2,3,4,5\}$, while the second is the same as that for a Killing spinor on a Calabi-Yau twofold with tangent directions $\{4,5,6,7\}$. The similar result can be also achieved via technique of probe D5-brane: the BPS configuration requires $\tilde{\theta} = \pi/2$ in metric (2.8) [13] such that again only eight space-time dimensions are relevant. So that we will conveniently set $\tilde{\theta} = \pi/2$ in our following calculations.

3. Fluctuation solution

In this paper we would like to study 1-point and 2-point Green functions of the following dimension four operator

$$\mathcal{O}(x) = \mathrm{Tr}(D_\mu\Phi^\dagger D^\mu\Phi + 2\bar{\Psi}_A \not{D}\Psi^A) - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}, \qquad (3.1)$$

where $(A_\mu,\ \Psi_A,\ \Phi)$ forms a $\mathcal{N}=2$ vector supermultiplet, and Tr taken over SU(N) gauge group. The original $\mathcal{N}=2$ SYM lagrangian is just this operator multiplying $1/g_{\rm YM}^2$. Then it is unambiguous that the fluctuation of gauge coupling $1/g_{\rm YM}^2$ couples to this operator [18]. Therefore, in order to derive field equation of fluctuation field, we should first find which fields of h, x or k determine the gauge coupling of $\mathcal{N}=2$ SYM.

For achieving this purpose, we have to up-lift metric (2.2) to ten dimensions without considering any special solutions. In string frame, the 10-d metric and dilation ω for D5-brane configuration are given by

$$ds_{10}^2 = e^\omega[dx_{1,3}^2 + dr^2 + r_0^2 e^{2h} d\Omega_2^2] + \cdots \,, \\ e^\omega = e^{5f}\Delta \,, \qquad (3.2)$$

where Δ has been defined in eq. (2.9). The Dirac-Born-Infeld (DBI) action for D5-brane is

$$\mathcal{L}_{\rm DBI} = -\tau_5 \int d^6\xi e^{-\omega}\sqrt{-\det(G + 2\pi\alpha' F)} + \cdots \,, \\ \tau_5 = (2\pi)^{-5} g_s^{-1}\alpha'^{-3} \,, \qquad (3.3)$$

where G and F are pull-back of 10-d metric and gauge fields. Conveniently, we can parameterize brane world-volume coordinates by $\xi = \{x^0, \ldots, x^3, \theta, \varphi\}$. Then integrating the

JHEP10(2002)005

compact part of D5-brane, we will achieve 4-d $\mathcal{N}=2$ SYM theory at leading order of α' expansion. The gauge coupling of $\mathcal{N}=2$ SYM is given by

$$\begin{aligned}\frac{1}{g_{\rm YM}^2} &= \frac{\tau_5(2\pi)^2\alpha'^2}{2}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta e^{-3\omega}\sqrt{-\det G}\,,\\ &= \frac{N}{4\pi^2}e^{2h}\,.\end{aligned}\tag{3.4}$$

This result is the same as one obtained in ref. [13]. Consequently, the β-function of $\mathcal{N}=2$ SYM can be achieved if we complement with the relation $z\sim\ln\mu/M$ as in ref. [13]. Now let

$$e^{2h}\;\rightarrow\;e^{2h(r)}+\left(\frac{N}{4\pi^2}\right)^{-1}\phi(x,r)=e^{2h(r)}+\tilde\phi(x,r)\,.\tag{3.5}$$

Thus field $\phi(x,r)$ is fluctuation of domain-wall solution, $z=e^{2h}$, and its boundary value $\phi(x,r_c)$ couples to operator $\mathcal{O}(x)$. It is in general consistent to consider the following ansatz

$$\tilde\phi(x,r)=s(r)\int\frac{d^4q}{(2\pi)^2}e^{iq\cdot x}\,.\tag{3.6}$$

Then inserting eq. (3.5) into 5-d action (2.3)[3] and expanding it up to quadratic terms of $\tilde\phi$, we get

$$\begin{aligned}\delta S_5 &= \delta S_5^{(1)}+\delta S_5^{(2)}\,,\\ \delta S_5^{(1)} &= \eta_5\int d^4x dr\; e^{2k-2h}\left\{2\frac{d^2h}{dr^2}+4\frac{dk}{dr}\frac{dh}{dr}-2r_0^{-2}e^{-2h}+r_0^{-2}e^{-4h-2x}\right\}\tilde\varphi-\\ &\quad -2\frac{d}{dr}\left(e^{2k-2h}\frac{dh}{dr}\tilde\phi\right),\\ \delta S_5^{(2)} &= \eta_5\int d^4x dr\; e^{2k-4h}\left\{-\frac{1}{2}\partial_\mu\tilde\phi\partial^\mu\tilde\phi-\frac{1}{2}\left(\frac{d\tilde\phi}{dr}\right)^2+2\frac{dh}{dr}\frac{d\tilde\phi^2}{dr}-M^2\tilde\phi^2\right\},\end{aligned}\tag{3.7}$$

where

$$M^2=6\left(\frac{dh}{dr}\right)^2-\frac{2}{r_0^2}e^{-2h}+\frac{3}{2r_0^2}e^{-4h-2x}-\frac{q^2}{2}.\tag{3.8}$$

The linear terms of $\tilde\phi$, i.e., $\delta S_5^{(1)}$ vanish due to field equation of h, up to a surface term.

The field equation yielded by action (3.7) is

$$\frac{d^2s}{dr^2}+2\frac{d(k-2h)}{dr}\frac{ds}{dr}+\left(q^2+4\left(\frac{dh}{dr}\right)^2-r_0^{-2}e^{-4h-2x}\right)s=0\,.\tag{3.9}$$

This equation does not possess analytic solution for complete expression of background (2.5) and (2.6). However, since we focus on divergent behavior and Green functions,

[3]Precisely, we should use full type IIB action instead of 5-d action because the duality is between a type IIB configuration and $\mathcal{N}=2$ SYM gauge theory. However, since both of background and fluctuation are independent of five transverse coordinates, the type IIB action can be compactified on $S^3\times S^2$. Then we can conveniently use this 5-d effective action and results must be the same as using full type IIB action.

JHEP10(2002)005

in fact we only need to know the behavior of $\phi(x,r)$ close to boundary. From metric (2.13) and (2.10) we see that there are two possible "boundaries" for this geometry when $\tilde{\theta} = \pi/2$,

$$\begin{cases} z \to \infty & \text{for all } c \\ z = 0 & \text{for } c < -1 \end{cases} \tag{3.10}$$

The condition $c < -1$ allows two boundaries existing simultaneously. In other words, D5-branes exhibit two flat pieces, corresponding to $z \to 0$ and $z \to \infty$ respectively. We are not interested on this case since it is ambiguous which boundary the gauge theory is defined onto. Then at near boundary limit $z \to \infty$, the domain-wall solution (2.5) and (2.6) is simplified,

$$\begin{aligned} e^{2x} &= 1\,, & e^{2k} &= ze^{2z}\,, \\ \frac{dk}{dr} &= -r_0^{-1}\left(1 + \frac{1}{4z}\right), & \frac{dh}{dr} &= -\frac{1}{2r_0 z}\,. \end{aligned} \tag{3.11}$$

As like as the expression of background (2.5), it is convenient to use new variable $z = e^{2h}$ to rewrite eq. (3.9). Then substituting this background into field equation (3.9), we have

$$\frac{d^2 s}{dz^2} + \left(2 - \frac{3}{2z}\right)\frac{ds}{dz} + r_0^2 q^2 s = 0\,. \tag{3.12}$$

This equation has two asymptotic solutions for large z:

$$\begin{aligned} &a) \quad s(z) \to z^{-m_1} e^{a_1 z}\,, \qquad a_1 = -1 + \sqrt{1 - r_0^2 q^2}\,, \qquad m_1 = \frac{3}{2}\frac{1 - \sqrt{1 - r_0^2 q^2}}{\sqrt{1 - r_0^2 q^2}}\,; \\ &b) \quad s(z) \to z^{m_2} e^{a_2 z}\,, \qquad a_2 = -1 - \sqrt{1 - r_0^2 q^2}\,, \qquad m_2 = \frac{3}{2}\frac{1 + \sqrt{1 - r_0^2 q^2}}{\sqrt{1 - r_0^2 q^2}}\,. \end{aligned} \tag{3.13}$$

Again, similar to AdS_5 case, the solution $a)$ is non-normalizable and $b)$ is normalizable. Since $r_0^2 q^2 \sim \alpha' q^2 \to 0$ for any low energy field theory, we further have

$$\begin{aligned} a_1 &= -\frac{1}{2} r_0^2 q^2\,, & m_1 &= \frac{3}{4} r_0^2 q^2\,, \\ a_2 &= -2 + \frac{1}{2} r_0^2 q^2\,, & m_2 &= 3 + \frac{3}{4} r_0^2 q^2\,. \end{aligned} \tag{3.14}$$

Imposing the boundary condition $\phi(x,z) = \phi_0(x) = e^{iq\cdot x}$ at $z = z_0 \to \infty$, we find the asymptotical solution of $\tilde{\phi}$ for large z as follows

$$\tilde{\phi}(x,z) \to \frac{\lambda_1 z^{-m_1} e^{a_1 z} + \lambda_2 z^{m_2} e^{a_2 z}}{\lambda_1 z_0^{-m_1} e^{a_1 z_0} + \lambda_2 z_0^{m_2} e^{a_2 z_0}} e^{iq\cdot x}\,, \tag{3.15}$$

where λ_1 and λ_2 are arbitrary constants.

4. Green functions and radial/energy-scale relation

4.1 Green functions

In SUGRA picture the one-point function of an operator corresponds to the first variation of the SUGRA action. In general, this quantity is expected to vanish due to field equation. However, the first variation is only required to vanish up to a total derivation term. Indeed,

JHEP10(2002)005

unlike conformal case, from action (3.7) we find that there is a total derivation term which belongs to linear term of $\tilde{\phi}$. It leads to one-point function of operator $\mathcal{O}(x)$ as follows [4]

$$\langle \mathcal{O}(x) \rangle = \left. \frac{\delta S}{\delta \phi_0(x)} \right|_{z=z_0,\phi_0=0} = r_0^{-1} \eta_5 \frac{e^{2z_0}}{z_0} \,. \tag{4.1}$$

In addition, two-point function is given by

$$\begin{aligned} \langle \mathcal{O}(p)\mathcal{O}(q) \rangle &= \left. \frac{\delta^2 S}{\delta \phi_0(p) \delta \phi_0(q)} \right|_{z=z_0,\phi_0=0} \\ &= 2\eta_5 \delta^4(p+q) e^{2k-4h} \left(-\frac{1}{2} s \frac{ds}{dr} + 2s^2 \frac{dh}{dr} \right) \Bigg|_{z=z_0} \\ &= \frac{\eta_5}{2r_0} \delta^4(p+q) \frac{e^{2z_0}}{z_0} \left[-\frac{r_0^2 q^2 z_0}{2} - \frac{3r_0^2 q^2}{4} - \frac{\lambda_2}{\lambda_1} (2 + r_0^2 q^2) z_0^{4+\frac{3r_0^2q^2}{2}} e^{-(2+r_0^2q^2)z_0} \right] + \\ &\quad + O(r_0^4 q^4) + O(e^{-4z_0}) \,. \end{aligned} \tag{4.2}$$

The result (4.2) is notable. In some terms momenta $r_0^2 q^2 \sim \alpha' q^2$ behaves exponentially, as like as Veneziano amplitude [20] at the hard scattering limit in perturbative string theory. It is obvious that this contribution can not be obtained from perturbative calculation of QFT. Precisely, it indicates that some non-local interactions are turned on. In other words, the operator $\mathcal{O}(x)$ describes a composite object which will be not pointlike at string scale. In next subsection we will show that, at leading order of α' expansion, it can be interpreted as non-perturbative instanton effect.

4.2 Radial/energy-scale relation

According to eqs. (2.12) and (3.4), the two-point function of $\mathcal{O}(x)$ can be re-expressed by radial parameter ρ and Yang-Mills coupling $g_{\rm YM}^2$,

$$\langle \mathcal{O}(p)\mathcal{O}(q) \rangle \sim -\eta_5 \frac{N g_{\rm YM}^2}{16\pi^2} \frac{\rho_0^2}{r_0} q^2 \delta^4(p+q) + O\left(\frac{\rho_0^2}{\ln \rho_0^2} \right) + O(g_{\rm YM}^4) \,, \tag{4.3}$$

where $\rho_0 \to \infty$ is large radial cut-off. In the above equation we have used the fact that large radial limit in SUGRA corresponds to weak 't Hooft coupling limit, $N g_{\rm YM}^2 \ll 1$, in SYM. It is same to AdS_5/CFT_4 case that, at the decoupling limit $\alpha' \to 0$, we expect a new radial variable $\nu = \frac{r_0^2}{\rho}$ should be independent of α'. Then we have

$$\langle \mathcal{O}(p)\mathcal{O}(q) \rangle \sim -r_0^3 \eta_5 \frac{N g_{\rm YM}^2}{16\pi^2} \epsilon^{-2} q^2 \delta^4(p+q) + O\left(\frac{1}{(\epsilon^2 \ln \epsilon^2)} \right) + O(g_{\rm YM}^4) \,, \tag{4.4}$$

where $\epsilon = \nu_0 \to 0$ also is a cut-off. Because $\eta_5 \sim \alpha'^{-3/2}$, this result is independent of α', same as the result of QFT. Furthermore, the perturbative calculation of QFT yields divergent structure of two-point function of $\mathcal{O}$ as follows

$$\langle \mathcal{O}(p)\mathcal{O}(q) \rangle \sim g_{\rm YM}^2 \left(\alpha_0 \Lambda^4 + \alpha_1 \Lambda^2 q^2 + O\left(\ln \frac{\Lambda^2}{q^2} \right) \right) \delta^4(p+q) \,. \tag{4.5}$$

[4]Here we again conveniently use effective 5-d action replacing full IIB action, since transverse part of IIB action can be integrated out before functional variations on $\tilde{\varphi}$.

Comparing q^2 terms in eqs. (4.4) and (4.5), we obtain $\epsilon \sim \Lambda^{-1}$. Here we have taken cut-off both for radial in gravity and for energy in QFT. If let them flow away from cut-off point, we have

$$\nu \sim \mu^{-1} \implies z = \ln \frac{\mu}{M} + C_0 \,, \tag{4.6}$$

where M is definite mass parameter and C_0 is an unimportant constant. This result precisely agrees with result in ref. [13].

The β-function of $\mathcal{N} = 2$ SYM can be directly obtained from eqs. (3.4) and (4.6),

$$\beta(g_{\rm YM}) = -\frac{1}{g_{\rm YM}^3} \frac{d}{d \ln(\mu/M)} \frac{1}{g_{\rm YM}^2} = -\frac{N}{8\pi^2} g_{\rm YM}^3 \,. \tag{4.7}$$

It precisely agrees with the result from perturbative calculation of QFT.

Now let us consider non-perturbative effect in two-point function (4.2). According to the identity

$$\lim_{z_0 \to \infty} z_0^3 e^{-(2+r_0^2 q^2) z_0} = \frac{6}{(2 + r_0^2 q^2)^3} e^{-(2+r_0^2 q^2) z_0} \,, \tag{4.8}$$

at $z_0 \to \infty$ and low energy limit $r_0^2 q^2 \to 0$ the q^2 terms of two-point can be written as

$$\langle \mathcal{O}(p) \mathcal{O}(q) \rangle \sim -r_0^3 \eta_5 \frac{N g_{\rm YM}^2}{16\pi^2} \left(\epsilon^{-2} - 6 r_0^{-2} \frac{\lambda_2}{\lambda_1} \right) q^2 \delta^4(p+q) \tag{4.9}$$

Comparing with the result of QFT, we achieve

$$\epsilon^{-2} - 6 r_0^{-2} \frac{\lambda_2}{\lambda_1} \sim \Lambda^2 \implies z = \frac{1}{2} \ln \left(\frac{\mu^2}{M^2} + 6 \frac{\lambda_2}{\lambda_1} \right) + C_0. \tag{4.10}$$

This radial/energy-scale relation leads β-function of $\mathcal{N} = 2$ SYM as follows

$$\beta(g_{\rm YM}) = -\frac{N}{8\pi^2} g_{\rm YM}^3 \left(1 + b \exp\left\{ -\frac{8\pi^2}{N g_{\rm YM}^2} \right\} + O\left(\exp\left\{ -\frac{16\pi^2}{N g_{\rm YM}^2} \right\} \right) \right) , \tag{4.11}$$

where b is an unknown constant. The extra term in the above expression can be treated as non-perturbative contribution from instantons with fractional charge k/N where k is a positive integer, as like as one has shown in pure $\mathcal{N} = 1$ SYM [21]. However, another possibility should also be considered: This extra terms may be artifact of the gravity solution that do not reproduce exactly $\mathcal{N} = 2$ SYM, but have extra KK modes, for example. If this were the case these extra terms are not gauge theory effects but just effects that are due to the UV completion, being a little string theory. In order to recognize the origination of these extra terms, more studies are needed.

4.3 Further discussions

The first notable fact is one-point function of $\mathcal{O}$. The non-zero value of this one-point function implies that the bosons in $\mathcal{N} = 2$ supermultiplet have condensation. The one-point function in eq. (4.1) can be rewritten as

$$\langle \mathcal{O}(x) \rangle = r_0^3 \eta_5 \frac{N g_{\rm YM}^2}{4\pi^2} \epsilon^{-4} \exp\left\{ -\frac{8\pi^2}{N g_{\rm YM}^2} \right\} . \tag{4.12}$$

This result matches with QFT calculation $\langle \mathcal{O} \rangle \sim \Lambda^4$. However, the exponential factor indicates that this condensation entirely is non-perturbative effect induced by fractional instantons.

It is well-known that the QFT description is reliable at weak 't Hooft coupling, $Ng_{\rm YM}^2 \ll 1$. However, the SUGRA description is reliable when radial is much large than string scale, i.e., $r_0^2/\alpha' \sim Ng_{\rm YM}^2 \gg 1$. So that SUGRA description on wrapped D5-brane configuration at weak coupling is dual to a strong coupled SYM in four dimensions, and vice versa. Thus it is not surprised why non-perturbative effects appear in Green functions. It is also same to our usual understanding on gauge/gravity duality.

In previous subsection the β-function for $\mathcal{N} = 2$ SYM is obtained directly when radial/energy-scale relation is imposed, even without considering any quantum corrections induced by usual quantum field theory method. It implies that some quantum effects have been included when we obtain $\mathcal{N} = 2$ SYM from wrapped D5-brane configuration. The essential reason is that the coupling of SYM obtained in this way is no longer constant, but is radial-dependent. Extremely, in refs. [13, 22] the authors showed that not only one-loop effects, but all of quantum effects is encoded in SYM. It is not surprised, since when we obtain a four-dimensional gauge theory via compactifing a six-dimensional theory in non-flat background, some higher dimensional gravity effects must be involved into four-dimensional theory. However, a confusion question appears: if we treat $\mathcal{N} = 2$ SYM as low energy theory of open string theory at defined limit, but without considering gauge/gravity correspondence, whether and/or how should we consider quantum corrections in this low energy theory? Whether is it double counting if we consider quantum correction? This confusion can be resolved when we are aware that the gauge theory is defined at the fixed boundary of space-time geometry. In this sense all couplings of SYM are still kept as constants at classical level, and the quantum correction can be consistent included via perturbative calculation of QFT but without any double counting. However, when we impose radial/energy-scale relation and let the definition of gauge theory flow away from boundary, in prior we have included the effects of gauge/gravity duality. Thus quantum effects are naturally yielded even without QFT calculation.

5. Conclusions

We have evaluated one-point and two-point Green functions of operator (3.1) in $\mathcal{N} = 2$ SYM according to gauge/gravity correspondence. The dual SUGRA describes a configuration that D5-branes wrap on supersymmetric 2-cycle. We analyzed divergence behavior of these Green functions and compared them with one obtained from perturbative calculation of QFT. Then we achieved radial/energy-scale relation for this $\mathcal{N} = 2$ gauge/gravity duality configuration. In terms of this relation, we obtained the β-function of $\mathcal{N} = 2$ SYM with fractional instanton contribution. All results match with one obtained from field theory. Perturbatively, our result also agrees with earlier result by authors of ref. [13]. We also showed that the bosons in $\mathcal{N} = 2$ supermultiplet have condensation due to instanton effects, which in general can not be observed in perturbative QFT.

Our investigation provides a principle method to study radial/energy-scale relation for any configuration on gauge/gravity correspondence. However, we will meet a technical difficulty when we generalize it to $\mathcal{N}=1$ case and want to re-examine $\mathcal{N}=1$ radial/energy-scale relation in ref. [13, 22]. That the gauge coupling in $\mathcal{N}=2$ case is determined by a single background field (see eq. (3.4)), but in $\mathcal{N}=1$ it is expressed by nonlinear combination of several background fields [13, 22]. Thus it is difficult to derive field equation of fluctuation of gauge coupling. To overcome this difficulty will be important to study radial/energy-scale relation for any gauge/gravity correspondence.

Acknowledgments

We thank Prof. E. Imeroni, J.-X. Lu and Ch.J. Zhu for useful discussions and comments.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S. Kachru and E. Silverstein, *4d conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183];
A.E. Lawrence, N. Nekrasov and C. Vafa, *On conformal field theories in four dimensions*, *Nucl. Phys.* **B 533** (1998) 199 [hep-th/9803015].

[3] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080];
K. Oh and R. Tatar, *Renormalization group flows on D3-branes at an orbifolded conifold*, *J. High Energy Phys.* **05** (2000) 030 [hep-th/0003183];
I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and chisb-resolution of naked singularities*, *J. High Energy Phys.* **08** (2000) 052 [hep-th/0007191].

[4] D.-E. Diaconescu, M.R. Douglas and J. Gomis, *Fractional branes and wrapped branes*, *J. High Energy Phys.* **02** (1998) 013 [hep-th/9712230];
N.A. Nekrasov, I.R. Klebanov *Gravity duals of fractional branes and logarithmic RG flow,Nucl. Phys.* **B 574** (2000) 263 [hep-th/9911096].

[5] M. Bertolini, P. Di Vecchia and R. Marotta, $N=2$ *four-dimensional gauge theories from fractional branes*, hep-th/0112195;
C.P. Herzog, I.R. Klebanov and P. Ouyang, *D-branes on the conifold and* $N=1$ *gauge/gravity dualities*, hep-th/0205100.

[6] M. Bershadsky, C. Vafa and V. Sadov, *D-branes and topological field theories*, *Nucl. Phys.* **B 463** (1996) 420 [hep-th/9511222].

[7] J.M. Maldacena and C. Núñez, *Towards the large-N limit of* $\mathcal{N}=1$ *super Yang-Mills*, *Phys. Rev. Lett.* **86** (2001) 588 [hep-th/0008001]

[8] P. Gauntlett, N. Kim, D. Martelli and D. Waldram, *Wrapped fivebranes and* $N=2$ *super Yang-Mills theory*, *Phys. Rev.* **D 64** (2001) 106008 [hep-th/0106117].

[9] F. Bigazzi, A.L. Cotrone and A. Zaffaroni, $N = 2$ *gauge theories from wrapped five-branes*, *Phys. Lett.* **B 519** (2001) 269 [hep-th/0106160];
R. Apreda, F. Bigazzi, A.L. Cotrone, M. Petrini and A. Zaffaroni, *Some comments on* $N = 1$ *gauge theories from wrapped branes*, *Phys. Lett.* **B 536** (2002) 161 [hep-th/0112236].

[10] Y. Kinar, A. Loewy, E. Schreiber, J. Sonnenschein and S. Yankielowicz, *Supergravity and world volume physics in the dual description of* $N = 1^*$ *theory*, *J. High Energy Phys.* **03** (2001) 013 [hep-th/0008141];
B.S. Acharya, J.P. Gauntlett and N. Kim, *Fivebranes wrapped on associative three-cycles*, *Phys. Rev.* **D 63** (2001) 106003 [hep-th/0011190];
H. Nieder and Y. Oz, *Supergravity and D-branes wrapping special lagrangian cycles*, *J. High Energy Phys.* **03** (2001) 008 [hep-th/0011288];
J. Gomis, *D-branes, holonomy and M-theory*, *Nucl. Phys.* **B 606** (2001) 3 [hep-th/0103115];
G. Papadopoulos and A.A. Tseytlin, *Complex geometry of conifolds and 5-brane wrapped on 2- sphere*, *Class. and Quant. Grav.* **18** (2001) 1333 [hep-th/0012034];
J.P. Gauntlett, N. Kim and D. Waldram, *M-fivebranes wrapped on supersymmetric cycles*, *Phys. Rev.* **D 63** (2001) 126001 [hep-th/0012195];
C. Núñez, I.Y. Park, M. Schvellinger and T.A. Tran, *Supergravity duals of gauge theories from F(4) gauged supergravity in six dimensions*, *J. High Energy Phys.* **04** (2001) 025 [hep-th/0103080];
M. Schvellinger and T.A. Tran, *Supergravity duals of gauge field theories from* $\mathrm{SU}(2) \times \mathrm{U}(1)$ *gauged supergravity in five dimensions*, *J. High Energy Phys.* **06** (2001) 025 [hep-th/0105019];
J. Gomis and J.G. Russo, $D = 2 + 1$ $N = 2$ *Yang-Mills theory from wrapped branes*, *J. High Energy Phys.* **10** (2001) 028 [hep-th/0109177];
R. Hernandez, *Branes wrapped on coassociative cycles*, *Phys. Lett.* **B 521** (2001) 371 [hep-th/0106055];
J. Gomis and T. Mateos, *D6-branes wrapping kaehler four-cycles*, *Phys. Lett.* **B 524** (2002) 170 [hep-th/0108080];
J. Gomis, *On SUSY breaking and* χ*SB from string duals*, *Nucl. Phys.* **B 624** (2002) 181 [hep-th/0111060];
N. Evans, M. Petrini and A. Zaffaroni, *The gravity dual of softly broken* $N = 1$ *super Yang-Mills*, *J. High Energy Phys.* **06** (2002) 004 [hep-th/0203203];
U. Gursoy, C. Núñez and M. Schvellinger, *RG flows from* Spin(7)*, CY 4-fold and HK manifolds to AdS, Penrose limits and PP waves*, *J. High Energy Phys.* **06** (2002) 015 [hep-th/0203124].

[11] J. de Boer, E. Verlinde and H. Verlinde, *On the holographic renormalization group*, *J. High Energy Phys.* **08** (2000) 003 [hep-th/9912012].

[12] A.W. Peet and J. Polchinski, *UV/IR relations in AdS dynamics*, *Phys. Rev.* **D 59** (1999) 065011 [hep-th/9809022].

[13] P. Di Vecchia, A. Lerda and P. Merlatti, $N = 1$ *and* $N = 2$ *super Yang-Mills theories from wrapped branes*, hep-th/0205204.

[14] N. Seiberg, *Supersymmetry and nonperturbative beta functions*, *Phys. Lett.* **B 206** (1988) 75;
N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in* $N = 2$ *supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087]; *Monopoles, duality and chiral symmetry breaking in* $N = 2$ *supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484.

[15] A. Salam and E. Sezgin, SO(4) *gauging of $N = 2$ supergravity in seven-dimensions*, *Phys. Lett.* **B 126** (1983) 295.

[16] J.M. Maldacena and C. Núñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, *Int. J. Mod. Phys.* **A 16** (2001) 822 [hep-th/0007018].

[17] M. Cvetič, H. Lu and C.N. Pope, *Consistent Kaluza-Klein sphere reductions*, *Phys. Rev.* **D 62** (2000) 064028 [hep-th/0003286].

[18] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[19] P. Di Vecchia, H. Enger, E. Imeroni and E. Lozano-Tellechea, *Gauge theories from wrapped and fractional branes*, *Nucl. Phys.* **B 631** (2002) 95 [hep-th/0112126].

[20] G. Veneziano, *An introduction to dual models of strong interactions and their physical motivations*, *Phys. Rept.* **9** (1974) 199.

[21] N.M. Davies, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Gluino condensate and magnetic monopoles in supersymmetric gluodynamics*, *Nucl. Phys.* **B 559** (199) 123 [hep-th/9905015].

[22] E. Imeroni, *On the $N = 1$ beta-function from the conifold*, *Phys. Lett.* **B 541** (2002) 189 [hep-th/0205216].

JHEP10(2002)005